

# Vertex identifying codes for the $n$ -dimensional lattice

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## Abstract

An  $r$ -identifying code on a graph  $G$  is a set  $C \subset V(G)$  such that for every vertex in  $V(G)$ , the intersection of the radius- $r$  closed neighborhood with  $C$  is nonempty and different. Here, we provide an overview on codes for the  $n$ -dimensional lattice, discussing the case of 1-identifying codes, constructing a sparse code for the 4-dimensional lattice as well as showing that for fixed  $n$ , the minimum density of an  $r$ -identifying code is  $\Theta(1/r^{n-1})$ .

## 1 Introduction

Vertex identifying codes were introduced by Karpovsky, Chakrabarty, and Levitin in [7] as a way to help with fault diagnosis in multiprocessor computer systems. Amongst the many results in that paper, an interesting result is that if  $n = 2^k - 1$  for some integer  $k$ , we can find a code of optimal density for the  $n$ -dimensional lattice by using a Hamming code. Denote by  $\mathcal{D}(G, r)$  the minimum possible density of an  $r$ -identifying code for a graph  $G$ . Let  $L_n$

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denote the  $n$ -dimensional lattice. In [8], we present a slight generalization of this proof.

**Theorem 1 ([8])** *Let  $D$  be a dominating set for the  $n$ -dimensional hypercube, then  $\mathcal{D}(L_n, 1) \leq |D|/2^n$ .*

The proof of this comes from replacing Hamming Codes (which are already dominating sets) with the more general dominating sets to get bounds in the case that  $n \neq 2^k - 1$ . For small values of  $n$ , we use Table 6.1 of [4] to get good bounds in Figure 1.

	$\mathcal{D}(L_1, 1)$	=	$1/2$
	$\mathcal{D}(L_2, 1)$	=	$7/20$ <sup>[1]</sup>
	$\mathcal{D}(L_3, 1)$	=	$1/4$
$1/5$	$\leq \mathcal{D}(L_4, 1)$	$\leq$	$2/9$ <sup>[Theorem 4]</sup>
$1/6$	$\leq \mathcal{D}(L_5, 1)$	$\leq$	$7/32$ <sup>[Theorem 1]</sup>
$1/7$	$\leq \mathcal{D}(L_6, 1)$	$\leq$	$3/16$ <sup>[Theorem 1]</sup>
	$\mathcal{D}(L_7, 1)$	=	$1/8$
$1/9$	$\leq \mathcal{D}(L_8, 1)$	$\leq$	$1/8$
$1/10$	$\leq \mathcal{D}(L_9, 1)$	$\leq$	$31/256$ <sup>[Theorem 1]</sup>
$1/11$	$\leq \mathcal{D}(L_{10}, 1)$	$\leq$	$15/128$ <sup>[Theorem 1]</sup>

Figure 1: A table of bounds of densities of codes for small values of  $n$ . All bounds not cited are due to [7].

The result for  $L_4$  is proven in Section 2. For larger values of  $n$  we use this theorem in conjunction with a result of Kabatyanskiĭ and Panchenko[6] to get a good asymptotic bound.

**Corollary 2** *There is a constant  $b$  such that for sufficiently large  $n$ :*

$$\frac{1}{n+1} \leq \mathcal{D}(L_n, 1) \leq \left(1 + \frac{b \ln \ln n}{\ln n}\right) \frac{1}{n+1}.$$

Finally, in Section 3, we prove both an upper and lower bound for  $\mathcal{D}(L_n, r)$  and show:

**Theorem 3** For fixed  $n$ ,  $\mathcal{D}(L_n, r) = \Theta(1/r^{n-1})$  as  $r \rightarrow \infty$ .

Given a connected, undirected graph  $G = (V, E)$ , we define  $B_r(v)$ , called the ball of radius  $r$  centered at  $v$  to be

$$B_r(v) = \{u \in V(G) : d(u, v) \leq r\}.$$

We call any nonempty subset  $C$  of  $V(G)$  a *code* and its elements *codewords*. A code  $C$  is called *r-identifying* if it has the properties:

1.  $B_r(v) \cap C \neq \emptyset$  for all  $v$
2.  $B_r(u) \cap C \neq B_r(v) \cap C$ , for all  $u \neq v$

When  $r = 1$  we simply call  $C$  an identifying code. When  $C$  is understood, we define  $I_r(v) = I_r(v, C) = B_r(v) \cap C$ . We call  $I_r(v)$  the identifying set of  $v$ . If  $I_r(u) \neq I_r(v)$  for some  $u \neq v$ , then we say  $u$  and  $v$  are *distinguishable*. Otherwise, we say they are *indistinguishable*.

We formally define the  $n$ -dimensional lattice  $L_n = (V, E)$  where

$$V = \mathbb{Z}^n, \quad E = \left\{ \{(x_1, \dots, x_n), (y_1, \dots, y_n)\} : \sum_{i=1}^n |x_i - y_i| = 1 \right\}.$$

The density of a code  $C$  for a finite graph  $G$  is defined as  $|C|/|V(G)|$ . Let  $Q_m$  denote the set of vertices  $(x_1, \dots, x_n) \in \mathbb{Z}^n$  with  $|x_i| \leq m$  for all  $1 \leq i \leq n$ . We define the density  $D$  of a code  $C$  in  $L_n$  similarly to how it is defined in [3] by

$$D = \limsup_{m \rightarrow \infty} \frac{|C \cap Q_m|}{|Q_m|}.$$

## 2 The 4-dimensional case

The king grid,  $G_K$ , is defined to be the graph on vertex set  $\mathbb{Z} \times \mathbb{Z}$  with edge set  $E_K = \{\{u, v\} : u - v \in \{(0, \pm 1), (\pm 1, 0), (1, \pm 1), (-1, \pm 1)\}\}$ .

**Theorem 4**  $\mathcal{D}(L_4, 1) \leq 2/9$ .

**Proof.** The idea of our proof is to take our original copy of a code for the king grid and copy it to two-dimensional cross-sections of  $L_4$ —shifting it when “up and to the right” when moving in the  $x_3$  direction and “up and to the left” when moving in the  $x_4$  direction.

Let  $C$  the identifying code of density  $2/9$  for the king grid given by Cohen, Honkala, Lobstein and Zémor in [5]. For the remainder of this proof, let  $B_1^G(v)$  denote the ball of radius 1 in the graph  $G$  and likewise, let  $I_1^G(v)$  denote the identifying set of  $v$  in  $G$ .

Let  $v \in V(L_4)$ . Since  $(1, 0, 0, 0)$ ,  $(0, 1, 0, 0)$ ,  $(1, 1, 1, 0)$ , and  $(1, -1, 0, 1)$  are linearly independent, we may write uniquely write  $v = (x, y, 0, 0) + i(1, 1, 1, 0) + j(1, -1, 0, 1)$ . Next, we define  $\varphi : V(L_4) \rightarrow V(G_K)$  by  $\varphi(v) = (x, y)$  and then define

$$C' = \{v \in V(L_4) : \varphi(v) \in C\}.$$

Fixing,  $i$  and  $j$ , we see that  $C'$  consists of isomorphic copies of  $C$  and so  $C'$  has the same density as  $C$ .

It is easy to check that  $\varphi(B_1^{L_4}(v)) = B_1^{G_K}((x, y))$ . For instance,  $\varphi(v + (1, 0, 0, 0)) = (x + 1, y) \in B_1^{G_K}(v)$  and  $\varphi(v + (0, 0, 1, 0)) = \varphi((x - 1, y - 1, 0, 0) + (i + 1)(1, 1, 1, 0) + j(1, -1, 0, 1)) = (x - 1, y - 1) \in B_1^{G_K}(x, y)$ . This shows two things. First, each vertex has a non-empty set, since  $|I_1^{L_4}(v)| = |I_1^{G_K}(\varphi(v))| \geq 1$ . Secondly, it shows that if  $\varphi(u) \neq \varphi(v)$ , then  $u$  and  $v$  are distinguishable. Hence, we only need to distinguish between vertices where  $\varphi(u) = \varphi(v)$ .

Without loss of generality, let  $u = (x, y, 0, 0)$  and  $v = (x, y, 0, 0) + i(1, 1, 1, 0) + j(1, -1, 0, 1)$  and so

$$d(u, v) = |i + j| + |i - j| + |i| + |j|.$$

If  $i$  and  $j$  are both non-zero, then either  $|i + j|$  or  $|i - j|$  is at least 1 and so  $d(u, v) \geq 3$ . If  $j = 0$ , then  $d(u, v) = 3|i| \geq 3$  and likewise if  $i = 0$  then  $d(u, v) \geq 3|j| \geq 3$ . Since  $d(u, v) \geq 3$  in all cases, we only need to consider  $I_1^{L_n}(v)$ . It is nonempty and doesn't intersect with  $B_1^{L_n}(u)$ . Thus,  $u$  and  $v$  are distinguishable, completing the proof.  $\square$

### 3 General Bounds and Construction

We finally wish to produce some general bounds for  $r$ -identifying codes on the  $L_n$ . We start with a lower bound proof, in the style of Charon, Honkala, Hudry and Lobstein[2]. First, we define  $b_k^{(n)} = |B_k(v)|$  for  $v \in V(L_n)$ .

**Theorem 5** *The minimum density of an  $r$ -identifying code for  $L_n$  is at least*

$$\mathcal{D}(L_n, r) \geq \frac{(n-1)! \lceil \log_2(2n+1) \rceil}{2^{n+1} r^{n-1} + p_{n-2}(r)}$$

where  $p_{n-2}(r)$  is a polynomial in  $r$  of degree no more than  $n-2$ .

**Proof.** Let  $v \in V(L_n)$  and  $u_1, u_2, \dots, u_{2n}$  be its neighbors. If  $d(v, x) > r+1$ , then it is easy to see that  $d(u_i, x) \geq r+1$  for all  $i$ . Likewise, it is easy to check that if  $d(v, x) \leq r-1$ , then  $d(u_i, x) \leq r$  for all  $i$ . In other words, all vertices outside of  $B_{r+1}(v)$  are not in  $B_r(s)$  for any  $s \in S = \{v, u_1, u_2, \dots, u_{2n}\}$  and all vertices inside of  $B_{r-1}(v)$  are in  $B_r(s)$  for all  $s \in S$ .

Next, let  $C$  be an  $r$ -identifying code for  $L_n$ . For  $s, s' \in S$  with  $s \neq s'$ , we must have  $I_r(s) \triangle I_r(s') \subset B_{r+1}(v) \setminus B_{r-1}(v)$ . Let  $K(s) = I_r(s) \cap (B_{r+1}(v) \setminus B_{r-1}(v))$ . We claim for  $K(s) \neq K(s')$ . Suppose not. Then  $I_r(s) = K(s) \cup (C \cap B_{r-1}(v)) = I_r(s')$  and so they are not distinguishable. Hence,  $K(s)$  must be distinct for each  $s \in S$ . Since the minimum number of elements of a set to produce  $2n+1$  distinct subsets is  $\lceil \log_2(2n+1) \rceil$ , there must be  $\lceil \log_2(2n+1) \rceil$  codewords in  $B_{r+1}(v) \setminus B_{r-1}(v)$ . We refer to the methods used by Charon, Honkala, Hudry and Lobstein [2] to show this gives the lower bound:

$$\frac{\lceil \log_2(2n+1) \rceil}{b_{r+1}^{(n)} - b_{r-1}^{(n)}}.$$

It is easy to check that  $b_r^{(n)}$  is the number of solutions in integers to

$$|x_1| + |x_2| + \dots + |x_n| \leq r \tag{1}$$

and so  $b_{r+1}^{(n)} - b_{r-1}^{(n)}$  is the number of solutions to

$$|x_1| + |x_2| + \dots + |x_n| = k$$

where  $k = r$  or  $k = r+1$ . Since the number of solutions to  $x_1 + x_2 + \dots + x_n = k$  is known to be  $\binom{n+k-1}{n-1}$ , this gives us an upper bound

$$\begin{aligned} b_{r+1}^{(n)} - b_{r-1}^{(n)} &\leq 2^n \left( \binom{n+r-1}{n-1} + \binom{n+r}{n-1} \right) \\ &\leq 2^n \left( \frac{(r+n-1)^{n-1}}{(n-1)!} + \frac{(r+n)^{n-1}}{(n-1)!} \right) \\ &= \frac{2^{n+1}r^{n-1} + p_{n-2}(r)}{(n-1)!} \end{aligned}$$

which comes from choosing each term to be either positive or negative and then using a standard binomial inequality. Plugging this in gives us the result described in the theorem.  $\square$

**Theorem 6** *If  $n$  is odd,  $0 \leq k < n+1$ ,  $r \geq n+2$ , and  $r \equiv k \pmod{n+2}$  then*

$$\mathcal{D}(L_n, r) \leq \frac{(n+2)^{n-1}}{2^n(r-k)^{n-1}}.$$

*If  $n$  is even,  $0 \leq k < (n+2)/2$ ,  $r \geq (n+2)/2$ , and  $r \equiv k \pmod{(n+2)/2}$  then*

$$\mathcal{D}(L_n, r) \leq \frac{(n+2)^{n-1}}{2^n(r-k)^{n-1}}.$$

**Proof.** Let  $2r_0$  be divisible by  $n+2$  and let  $k = 2r_0/(n+2)$ . We wish to find an  $r$ -identifying code for  $r \geq r_0$ . We define a code

$$C = \{(kx_1, kx_2, \dots, kx_{n-1}, \ell) : x_1 + x_2 + \dots + x_{n-1} \equiv 1 \pmod{2}\}.$$

Further, let

$$S = \{(kx_1, kx_2, \dots, kx_{n-1}, \ell) : x_1 + x_2 + \dots + x_{n-1} \equiv 0 \pmod{2}\}.$$

$C$  will be our code and  $S$  will serve as a set of reference points which we will use later.

We first wish to calculate the density  $C \cup S$ . This is simply a tiling of  $\mathbb{Z}^n$  by the region  $[0, k-1]^{n-1} \times \{0\}$  which has only a single codeword in it. Hence, the density of  $C \cup S$  is  $1/k^{n-1} = (n+2)^{n-1}/(2^{n-1}r_0^{n-1})$ . Then  $C$  is half this density, which is the density stated in the theorem.

Next, we wish to show that  $C$  is an  $r$ -identifying code for  $r \geq r_0$ . Let  $e^{(i)}$  represent the vector with a 1 in the  $i$ th coordinate and a 0 in all other coordinates. For any vertex  $u$ , let  $u_j$  denote the value of the  $j$ th coordinate of  $u$ .

For  $s \in S$ , we define the corners of  $s$  to be the codewords  $c$  of the form  $c = s \pm ke^{(i)}$  for some  $1 \leq i \leq n-1$ .

The remainder of the proof consists of 3 steps:

1. Each vertex  $v \in V(G)$  has distance at most  $nk/2$  from some  $s \in S$  and  $v$  has distance at most  $r$  from each of the corners of  $s$  (in addition, this shows that  $I_r(v)$  is nonempty).
2. If  $v = (\mathbf{v}, \ell)$ , we can uniquely determine  $\ell$  from  $I_r(v)$ . Furthermore, if  $c = (\mathbf{c}, \ell) \in I_r(v)$ , we can determine  $d(v, c)$ .
3. If  $v = (v_1, \dots, v_{n-1}, \ell)$ , we can uniquely determine  $v_i$  from  $I_r(v)$  for each  $i$ . Thus,  $v$  is distinguishable from all other vertices in the graph.

**Step 1:** Let  $v = (v_1, v_2, \dots, v_{n-1}, \ell)$ . Without loss of generality, we may assume that  $(v_1, v_2, \dots, v_{n-1}) \in [0, k]^{n-1}$ . For  $i = 1, 2, \dots, n-2$  define

$$a_i = \begin{cases} 0 & \text{if } v_i \leq k/2 \\ k & \text{if } v_i > k/2 \end{cases}.$$

We then see that  $|v_i - a_i| \leq k/2$  in either case. Now consider the vertices  $(a_1, a_2, \dots, a_{n-2}, 0, \ell)$  and  $(a_1, a_2, \dots, a_{n-2}, k, \ell)$ . One of these is in  $S$ . Let  $a_{n-1} = 0$  if the former is in  $S$  and  $a_{n-1} = k$  if the latter is in  $S$ . Then  $|v_{n-1} - a_{n-1}| \leq k$ . Hence we have

$$\begin{aligned} d(v, (a_1, a_2, \dots, a_{n-2}, a_{n-1}, \ell)) &= |v_{n-1} - a_{n-1}| + \sum_{i=1}^{n-2} |v_i - a_i| \\ &\leq k + (n-2)k/2 = nk/2. \end{aligned}$$

Let  $c$  be a corner of  $s = (a_1, a_2, \dots, a_{n-2}, a_{n-1}, \ell)$ . Then

$$d(v, c) \leq d(v, s) + d(s, c) \leq nk/2 + k = (n+2)k/2 = r_0 \leq r.$$

**Step 2:** Next, we need to determine the last coordinate of  $v$ . Write  $v = (\mathbf{v}, \ell)$ . Suppose that  $c = (\mathbf{c}, k) \in I_r(v)$ . We then see that  $(\mathbf{c}, \ell) \in I(v)$  since  $d(v, (\mathbf{c}, \ell)) \leq d(v, c)$ . Writing  $d(v, (\mathbf{c}, \ell)) = d_1 \leq r$ , then we see that  $(\mathbf{c}, \ell \pm j) \in I(v)$  for  $j = 0, 1, \dots, r - d_1$ . Hence, these codewords form a path of length  $2(r - d_1) + 1$ . Thus, if  $\ell_1 = \min\{j : (\mathbf{c}, j) \in I(v)\}$  and  $\ell_2 = \max\{j : (\mathbf{c}, j) \in I(v)\}$ , it follows that

$$\ell = \frac{\ell_1 + \ell_2}{2}.$$

Furthermore, this tells us once we know  $\ell$ , we can determine the distance between  $v$  and  $c$  to be  $r - (\ell_2 - \ell)$ .

**Step 3:** Finally, from Step 1 we know that there is some vertex  $s \in S$  such that the codewords  $s \pm ke^{(i)} \in I_r(v)$  for each  $i$ ,  $1 \leq i \leq n - 1$ . Thus, for each  $i$  we are guaranteed that there are  $m \geq 2$  codewords  $c^{(0)}, \dots, c^{(m-1)}$  such that  $c^{(j)} = c^{(0)} + 2kje^{(i)}$  and  $c^{(j)} \in I_r(v)$  for each  $j$ .

Now let

$$D = \sum_{\substack{p=1 \\ p \neq i}}^{n-1} |v_p - c_p^{(0)}|.$$

We then see that  $d(v, c^{(j)}) = |v_i - c_i^{(j)}| + D$  which is minimized by minimizing  $|v_i - c_i^{(j)}|$ . Furthermore, the expression  $|v_i - x|$  is unimodal and so the two smallest values of  $|v_i - c_i^{(j)}|$  must happen for consecutive integers and they must be amongst our aforementioned  $m$  codewords. Let  $a = c^{(\ell)}$  and  $b = c^{(\ell+1)}$  be these codewords. It is easy to check that  $a_i \leq v_i \leq b_i$  by considering evenly spaced point plotted along the graph of  $f(x) = |v_i - x|$ .

This gives

$$\begin{aligned} d(v, a) &= v_i - a_i + D \\ d(v, b) &= b_i - v_i + D \end{aligned}$$

Since  $a$  and  $b$  are codewords, the distances listed above are all known quantities from Step 2. Subtracting the second line from the first and solving for  $v_i$  gives:

$$v_i = \frac{d(v, a) - d(v, b) + a_i + b_i}{2}.$$

Since these are all known quantities, we can compute  $v_i$ , completing step 3. Finally, we get the values described in the theorem by taking  $r_0$  to be the



largest integer smaller than  $r$  satisfying the condition that  $2r_0/(n+2)$  is an integer, completing the proof.  $\square$

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## 5 Conclusions

It is worth noting that the lower bound given in Theorem ?? can only be evaluated as  $r \rightarrow \infty$  and not as  $n \rightarrow \infty$  since the polynomial in the denominator is a polynomial in  $r$ , but the coefficients depend on  $n$ . However, for fixed  $n$  we can make a comparison of the bounds by taking the ratio of the upper bound to the lower bound. This gives:

$$\begin{aligned}
& \frac{(n+1)^{n-1}}{2^n r^{n-1}} \bigg/ \frac{(n-1)! \lceil \log_2(2n+1) \rceil}{2^{n+1} r^{n-1} + o(r^{n-1})} \\
&= \frac{2^{n+1} r^{n-1} + o(r^{n-1})}{2^n r^{n-1}} \cdot \frac{(n+1)^{n-1}}{(n-1)! \lceil \log_2(2n+1) \rceil} \\
&\approx (2 + o(1)) \cdot \frac{(n+1)^{n-1}}{(n-1)^{n-1}} \cdot \frac{e^{n-1}}{\sqrt{2\pi n} \lceil \log_2(2n+1) \rceil} \\
&\approx \frac{2e^{n+1}}{\sqrt{2\pi n} \lceil \log_2(2n+1) \rceil}
\end{aligned}$$

and so our lower bound differs from our upper bound by slightly less than a multiplicative factor of  $e^n$  when  $r \gg n \gg 0$ .

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